

Lively quantum walks on cycles

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Abstract

We introduce a family of quantum walks on cycles parametrized by their liveliness, defined as the ability to execute a long-range move. We investigate the behavior of the probability distribution and time-averaged probability distribution. We show that the liveliness parameter has a direct impact on the periodicity of the limiting distribution. We also show that the introduced model provides a simple recipe for improving the efficiency of the network exploration.

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1 Introduction

Quantum walks [1–3], quantum counterparts of classical Markov processes, provide a powerful method for developing new quantum algorithms [4] and protocols [5–10]. As quantum protocols have to be executed on pair with classical protocols controlling distant parts of a quantum network, quantum walks have to include elements enabling them to adapt to the current structure of the network. The methods of adapting classical algorithms for the purpose of quantum networks are currently under an active investigation [11] and include the application of game theory in a complex quantum network with interacting parties [12].

Quantum walks on cycles can be used as a simple and very powerful model for the purpose of modeling quantum and hybrid classical-quantum networks. In particular, in [8], the authors have developed a model that can be used to analyze the scenario of exploring quantum networks with a distracted sense of direction. By using this model, it is possible to study the behavior of quantum mobile agents operating with non-adaptive and adaptive strategies that can be employed in this scenario.

The presented work introduces a family of quantum walks on cycles parametrized by their *liveliness*, *i.e.* the ability to execute a long-range move. In particular, the proposed family contains lazy quantum walks, which can be introduced as quantum walks with liveliness equal to 0. We investigate the behavior of the probability distribution and time-averaged probability distribution [13] for the introduced family and generalize the results obtained by Bednarska *et*

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al. [14]. We show that the liveliness parameter has a direct impact on the periodicity of the limiting distribution. We also show that the introduced model provides a simple recipe for improving the efficiency of network exploration.

This paper is organized as follows. In Section 2 we introduce the model of lively quantum walks on cycles. In Section 3 we study the behavior of the time-averaged limiting distribution of the introduced model and discuss its periodicity. In Section 4 we prove that the introduced model allows the improvement of the quantum network exploration. We demonstrate that it can be used to avoid trapping and to detect malfunctions in the network. Finally, in Section 5, we discuss the possible applications and extensions of the introduced model.

2 Lively quantum walks

Let us consider a cycle with n nodes. The position of a walker during a quantum walk executed on such cycle is described by a vector in n -dimensional complex space \mathbb{C}^n . Let us assume that the coin register used to control a quantum walker is represented by a vector in \mathbb{C}^3 (i.e. by a qutrit) and thus, during each step, the walker can change its position according to one of three possible states of the coin register. Using this setup we define *lively quantum walk* on cycles as follows.

Definition 1 (Lively quantum walk on a cycle) *Let us denote by $|\psi\rangle = |c\rangle \otimes |x\rangle$ the state of the quantum walk with $|c\rangle \in \mathbb{C}^3$ and $|x\rangle \in \mathbb{C}^n$. Lively quantum walk on a n -dimensional cycle with liveliness $0 \leq a \leq \lfloor \frac{n}{2} \rfloor$, is defined by the shift operator of the form*

$$S = \sum S_x^{(n,a)}, \quad (1)$$

where

$$\begin{aligned} S_x^{(n,a)} = & |0\rangle\langle 0| \otimes |x-1 \pmod n\rangle\langle x| \\ & + |1\rangle\langle 1| \otimes |x+1 \pmod n\rangle\langle x| \\ & + |2\rangle\langle 2| \otimes |x+a \pmod n\rangle\langle x|. \end{aligned} \quad (2)$$

For the case $a = 0$ the above definition reduces to *lazy quantum walk* (i.e. a quantum walk with no liveliness). The existence of the additional connections can be used to model the transition from a cycle to the full network. For example, for $n = 6$, the lively quantum walk with $a = 2$ is equivalent to the quantum walk on the total network.

Operator $S_x^{(n,a)}$ acts on position x by shifting it by $+1$, -1 or by the value specified by the liveliness parameter $a \in \{0, 1, \dots, n-1\}$. The case $a = 0$ is identical to the case $a = n$.

Usually we start with a special initial state of the coin register, which is symmetric in some sense. Below we use

$$|c_1\rangle = \frac{1}{\sqrt{3}} (|0\rangle + |1\rangle + |2\rangle) \quad (3)$$

or

$$|c_2\rangle = \frac{1}{\sqrt{3}} (|0\rangle + i|1\rangle - |2\rangle). \quad (4)$$

The state register is initiated in a flat superposition of the base states

$$|x_0\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} |k\rangle. \quad (5)$$

The coin operator used in the further considerations is defined by the Grover operator

$$G = 2|c_1\rangle\langle c_1| - \mathbb{1}_3, \quad (6)$$

which translates into a matrix form

$$G = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}. \quad (7)$$

Using the above we define the walk operator for the lively walk on cycle as

$$\left(\sum_{x=0}^{n-1} S_x^{(n,a)} \right) (G \otimes \mathbb{1}_n). \quad (8)$$

3 Limiting distribution periodicity

We start with the proof of the periodicity of the limiting distribution for the introduced model. Let us introduce the time-averaged probability distribution for the quantum walk as follows.

Definition 2 *We define the time-averaged probability distribution at position x as*

$$\Pi(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N P(x, t), \quad (9)$$

where $P(x, t)$ denotes the probability of measuring position x after t steps

$$P(x, t) = \sum_c |\langle c, x | U^t | \psi_0 \rangle|^2. \quad (10)$$

Let us now consider a lively quantum walk with n nodes and the step size a chosen in such a way that there is a common divisor of both numbers.

Theorem 1 *If $\text{GCD}(a, n) > 1$ then the limiting time-averaged probability distribution is periodic with period equal to $\text{GCD}(a, n)$.*

First, let us note that the spectrum of the walk operator is conveniently expressed using Fourier basis at the position register.

Lemma 1 *The composite operator $U = S^{(n,a)}G \in L(\mathbb{C}^3, \mathbb{C}^n)$ has eigenvalues $\lambda(k, j)$ with corresponding eigenvectors $|\psi_{k,j}\rangle = |v(k, j)\rangle \otimes |\phi_k\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^n$ satisfying the equation*

$$\overline{U(k)}|v(k, j)\rangle = \text{diag}(e^{-ik}, e^{ik}, e^{-ika})G|v(k, j)\rangle = \lambda(k, j)|v(k, j)\rangle, \quad (11)$$

where $|\phi_k\rangle = \sum e^{-ikx}|x\rangle$ for $k = \frac{2\pi l}{n}, l = 0, \dots, n-1, j \in \{0, 1, 2\}$.

Proof. We analyse the action of the step on $|d\rangle \otimes |\phi_k\rangle$ for basis state $d \in \{\rightarrow, \leftarrow, \nearrow\}$ and note that the step operator acting on states with Fourier states on position register results in a relative phase $S^{(n,a)}|\leftarrow\rangle \otimes |\phi_k\rangle = e^{ik}|\leftarrow\rangle \otimes |\phi_k\rangle$, $S^{(n,a)}|\rightarrow\rangle \otimes |\phi_k\rangle = e^{-ik}|\rightarrow\rangle \otimes |\phi_k\rangle$ and $S^{(n,a)}|\nearrow\rangle \otimes |\phi_k\rangle = e^{-ika}|\nearrow\rangle \otimes |\phi_k\rangle$. Thus we can reduce the dynamics of the states of the form $|v\rangle \otimes |\phi_k\rangle$ so that we consider the $|v\rangle$ subsystem only and we substitute the step operator with $\text{diag}(e^{-ik}, e^{ik}, e^{-ika})$. Therefore any eigenvector $|v(k, j)\rangle$ of the form given in Eq. (11) corresponds to an eigenvector of the form $|v(k, j)\rangle \otimes |\phi_k\rangle$ of the walk operator U . ■

In the context of the limiting distribution we emphasise the fact that for any eigenvector of the form $|\psi_{k,j}\rangle = |v(k,j)\rangle \otimes |\psi_k\rangle$ the probability distribution

$$P_{k,j}(x) = \{(|\langle d| \otimes \langle x|)(|v(k,j)\rangle \otimes |\psi_k\rangle)|^2\}_d = \{|\langle d|v(k,j)\rangle|^2\}_d \quad (12)$$

over directions is position independent. This property can be applied into limiting distribution formula

$$\Pi(x) = \sum_{\lambda} \sum_d \sum_{(k,j),(k',j') \in V_{\lambda}} a_{k,j} a_{k',j'}^{\dagger} \langle d, x | \psi_{k,j} \rangle \langle \psi_{k',j'} | d, x \rangle, \quad (13)$$

where x is the position, $|\psi_0\rangle = \sum a_{k,l} |\psi_{k,l}\rangle$, V_{λ} are indices of λ -eigenvectors such that $V_{\lambda} = \{(k,j) : \lambda_{k,j} = \lambda\}$. It is straightforward from Eq. 12 that for 1-dimensional eigenspaces the probability is transition invariant

$$a_{k,j} a_{k,j}^{\dagger} \langle d, x | \psi_{k,j} \rangle \langle \psi_{k,j} | d, x \rangle = a_{k,j} a_{k,j}^{\dagger} \langle d, x' | \psi_{k,j} \rangle \langle \psi_{k,j} | d, x' \rangle, \quad (14)$$

for $x, x' = 1, \dots, n$. For higher-dimensional eigenspaces we are concerned with relative phase during transition. In other words, one is assured that for two positions x, x' the modules of the terms $\langle d, x | \psi_i \rangle \langle \psi_j | d, x \rangle$ and $\langle d, x' | \psi_i \rangle \langle \psi_j | d, x' \rangle$ are equal, however they may differ in phase. Here we prove that the dimensionality of eigenspaces of eigenvalues $\lambda_{k,j}$ is higher than one if the relation $k = \frac{2\pi l}{n}$ is satisfied for l being the multiplication of $\frac{n}{\text{GCD}(n,a)}$. We do not prove that for $k \neq \frac{2\pi l}{\text{GCD}(n,a)}$ the $\lambda_{k,j}$ -eigenspace is one-dimensional, but the influence of the cases when it is not true is negligible.

Lemma 2 For $k = \frac{2\pi l}{n}$ and $\frac{n}{\text{GCD}(n,a)} | l$ we have that $\lambda(k,0) = 1$ is an eigenvalue of $\overline{U(k)}$ and the other eigenvalues $\lambda(k,1)$, $\lambda(k,2)$ are mutually conjugated i.e. equal to $e^{\pm i\varphi_k}$. Moreover, eigenvalues for $k' = 2\pi - k$ are the same.

Proof. Let us derive the characteristic polynomial for eigenvalues of the step operator

$$\overline{U(k)} = G \text{diag}(e^{-ik}, e^{ik}, e^{-ika}). \quad (15)$$

We use the fact that $e^{-ika} = 1$. Thus the characteristic polynomial simplifies to

$$(1 - \lambda)(1/3\lambda(1 - 2\cos\Phi_k) + 1 + \lambda + \lambda^2), \quad (16)$$

with real coefficients and the same solution for $-k$ thus $\lambda_0 = 1$ and lemma holds. The explicit formulas for the eigenvalues with substitution for $e^{ik} = \omega$ are

$$\left\{ 1, -\frac{\omega^2 + 4\omega + (\omega - 1)\sqrt{\omega(\omega + 10) + 1} + 1}{6\omega}, -\frac{\omega^2 + 4\omega - (\omega - 1)\sqrt{\omega(\omega + 10) + 1} + 1}{6\omega} \right\}$$

and eigenvectors without normalization factors read

$$\begin{pmatrix} \frac{\omega+1}{2} & \omega & 1 \\ \frac{1}{2} \left(\omega + \sqrt{\omega(\omega + 10) + 1} + 1 \right) & \frac{\omega^2 \left(\omega + \sqrt{\omega(\omega + 10) + 1} + 5 \right)}{-5\omega + \sqrt{\omega(\omega + 10) + 1} - 1} & 1 \\ \frac{1}{2} \left(\omega - \sqrt{\omega(\omega + 10) + 1} + 1 \right) & \frac{\omega^2 \left(-\omega + \sqrt{\omega(\omega + 10) + 1} - 5 \right)}{5\omega + \sqrt{\omega(\omega + 10) + 1} + 1} & 1 \end{pmatrix} \quad (17)$$

■

Proof of Theorem 1. In order to prove the cyclic property of the walk we consider the following limiting distribution formula

$$\Pi(x) = \sum_{\lambda} \sum_{(k,l),(k',l') \in V_{\lambda}, d} a_{k,l}^{\dagger} a_{k',l'}^{\dagger} \langle d, x | \psi_{k,l} \rangle \langle \psi_{k',l'} | d, x \rangle. \quad (18)$$

We aim at proving that $\Pi(x) = \Pi(x+q)$, $q = \text{GCD}(n, a)$. We note that for one-dimensional eigenspaces the result $|\langle d, x | \psi_{k,l} \rangle|^2 = |\langle d, x+a | \psi_{k,l} \rangle|^2$ is straightforward from Lemma 1. We assume that multiple eigenvalues $\lambda_{k,j} = \lambda_{k',j'}$ follow the case $k = \frac{2nl}{\text{GCD}(a,n)}$, $k' = \frac{2n(n-l)}{\text{GCD}(a,n)}$ resulting from construction in Lemma 2 and give the $e^{ika} = e^{ik'a} = 1$. In particular, we show that the relative phases that occur in the terms of the sum Eq. (18) vanish every a steps. Thus the equality

$$\langle d, x+a | \Phi_{k,j} \rangle \langle \Phi_{k',j'} | d, x+a \rangle = \langle d, x | \Phi_{k,j} \rangle \langle \Phi_{k',j'} | d, x \rangle \quad (19)$$

holds for $(k,l), (k',l') \in V_{\lambda}$ and overall limiting probability is periodic with the period equal to $\text{GCD}(a, n)$. ■

Thus, we observe interesting phenomena that if the paths generated by the lively steps do not interact *i.e.* create separate classes of nodes, then the asymptotic probabilities become equal within these classes. This behavior is illustrated in Fig. 1.

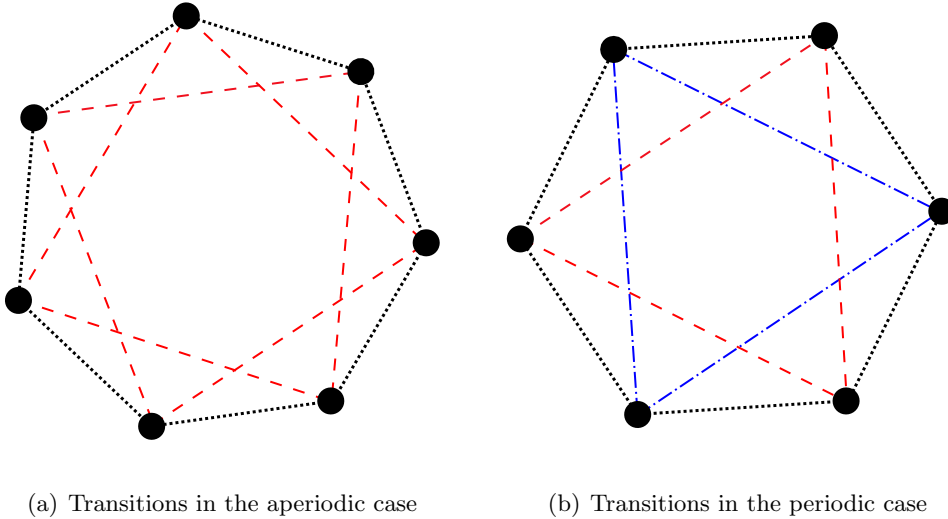


Figure 1: Illustration of the simple networks for the lively walk with $n = 6$ and $n = 7$ nodes and the liveliness $a = 2$. In the case (a) all nodes belong to the same class. In the case (b) red (dashed) and blue (dot-dashed) links connect the nodes corresponding to different classes exhibiting different asymptotic behavior.

4 Liveliness of lively walks

4.1 Mean difference of position

In this section we show that the name of the utilization of the introduced model is motivated by the ability to avoid the trapping. To be more precise we prove the following theorem.

Theorem 2 Let $\mathcal{H}_C = \mathbb{C}^3$, $\mathcal{H}_P = \mathbb{C}^n$ be Hilbert spaces and $\mathcal{D}(\mathcal{H}_C)$ and $\mathcal{D}(\mathcal{H}_P)$ spaces of density operators corresponding to them. Let us suppose that we have a lively walk with an initial state

$\rho_0 \in \mathcal{D}(\mathcal{H}_C \otimes \mathcal{H}_P)$ in the form $\rho_0 = \frac{1}{3}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|) \otimes |x\rangle\langle x|$. The shift operator is defined as in Eq. (2):

$$S = \sum_c |c\rangle\langle c| \otimes U_c, \quad (20)$$

where $c \in \{0, 1, 2\}$ and each pair of matrices U_c commute. Then for arbitrary three-dimensional coin operator the lively walk can not be trapped i.e. the expectation value of the difference of positions is non-zero.

Proof. Let $\{|e_m\rangle\}_{m=1}^n$ be eigenvectors of matrices U_c . First let us note that

$$\begin{aligned} \langle i | \text{tr}_P S \rho S^\dagger | j \rangle &= \sum_{c, c'} \langle i | \text{tr}_P (|c\rangle\langle c| \otimes U_c) \rho (|c'\rangle\langle c'| \otimes U_{c'}^\dagger) | j \rangle \\ &= \sum_{c, c', p} \langle p | (\langle i | \otimes \mathbb{1}_n) (|c\rangle\langle c| \otimes U_c) \rho (|c'\rangle\langle c'| \otimes U_{c'}^\dagger) (|j\rangle \otimes \mathbb{1}_n) | p \rangle \\ &= \sum_p \langle p | (\langle i | \otimes U_i) \rho (|j\rangle \otimes U_j^\dagger) | p \rangle \\ &= \sum_p \langle i | (\mathbb{1}_3 \otimes \langle p |) (\mathbb{1}_3 \otimes U_i) \rho (\mathbb{1}_3 \otimes U_j^\dagger) (\mathbb{1}_3 \otimes |p\rangle) | j \rangle \\ &= \langle i | \text{tr}_P (\mathbb{1}_3 \otimes U_i) \rho (\mathbb{1}_3 \otimes U_j^\dagger) | j \rangle. \end{aligned} \quad (21)$$

Additionally, for arbitrary operators $A \in \mathbb{C}^{n_1}$ and $B \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$, the partial trace is defined (see e.g. [15]) by the relation

$$\text{tr}(B(A \otimes \mathbb{1}_{n_2})) = \text{tr}(\text{tr}_2(B)A). \quad (22)$$

By mathematical induction we will show that below expressions are true for all $l \in \mathbb{N} \cup \{0\}$

$$\text{tr}_P(\rho_l) = \frac{\mathbb{1}_3}{3}, \quad (23)$$

$$\langle i, e_m | \rho_l | j, e_m \rangle = 0, m \in \{1, \dots, n\}, i, j \in \{0, 1, 2\}, i \neq j \quad (24)$$

$$\langle i, e_m | \rho_l | i, e_m \rangle = a_m, i \in \{0, 1, 2\}, m \in \{1, \dots, n\}, \quad (25)$$

where

$$\rho_l = (S \cdot (C \otimes \mathbb{1}_n))^l \rho_0 ((S \cdot (C \otimes \mathbb{1}_n))^\dagger)^l. \quad (26)$$

Also let us denote the state after the action of the coin operator and before the action of the shift operator

$$\rho_l^C = (C \otimes \mathbb{1}_n) \rho_l (C \otimes \mathbb{1}_n)^\dagger. \quad (27)$$

Let us show that the relations in Eqs. (24)-(25) are valid for an initial state. The overlap of ρ_0

$$\begin{aligned} \langle i, e_m | \rho_0 | j, e_m \rangle &= \langle i, e_m | \left(\frac{\mathbb{1}_3}{3} \otimes |x\rangle\langle x| \right) | j, e_m \rangle \\ &= \frac{1}{3} \langle i | j \rangle \otimes \langle e_m | x \rangle \langle x | e_m \rangle = 0, \end{aligned} \quad (28)$$

for $i \neq j$. Furthermore,

$$\text{tr}_P(\rho_0) = \text{tr}_P\left(\frac{\mathbb{1}_3}{3} \otimes |x\rangle\langle x|\right) = \frac{\mathbb{1}_3}{3}, \quad (29)$$

and

$$\begin{aligned}
\langle i, e_m | \rho_0 | i, e_m \rangle &= \langle i, e_m | \left(\frac{\mathbb{1}_3}{3} \otimes |x\rangle\langle x| \right) | i, e_m \rangle \\
&= \frac{1}{3} \langle i | \mathbb{1}_3 | i \rangle \otimes \langle e_m | x \rangle \langle x | e_m \rangle \\
&= \frac{1}{3} |\langle e_m | x \rangle|^2 = a_m.
\end{aligned} \tag{30}$$

Let us assume that the expressions (24, 23, 25) hold for some fixed $l \in \mathbb{N} \cup \{0\}$. We make the induction step for expression (23). First, we act with the coin operator

$$\begin{aligned}
\text{tr}_P(\rho_l^C) &= \sum_p (\mathbb{1}_3 \otimes \langle p |) (C \otimes \mathbb{1}_n) \rho_l (C^\dagger \otimes \mathbb{1}_n) (\mathbb{1}_3 \otimes |p\rangle) \\
&= \sum_p (C \otimes \mathbb{1}_n) (\mathbb{1}_3 \otimes \langle p |) \rho_l (\mathbb{1}_3 \otimes |p\rangle) (C^\dagger \otimes \mathbb{1}_n) \\
&= (C \otimes \mathbb{1}_n) \text{Tr}_P(\rho_l) (C^\dagger \otimes \mathbb{1}_n) = \frac{\mathbb{1}_3}{3},
\end{aligned} \tag{31}$$

and next we act with the shift operator

$$\begin{aligned}
\langle i | \text{tr}_P(S \rho_l^C S^\dagger) | i \rangle &= \langle i | \text{tr}_P(\mathbb{1}_3 \otimes U_i) \rho_l^C (\mathbb{1}_3 \otimes U_j^\dagger) | i \rangle \\
&= \text{tr}(\text{tr}_P((\mathbb{1}_3 \otimes U_i) \rho_l^C (\mathbb{1}_3 \otimes U_j^\dagger)) | i \rangle \langle i |) \\
&= \text{tr}(\mathbb{1}_3 \otimes U_i) \rho_l^C (\mathbb{1}_3 \otimes U_j^\dagger) (| i \rangle \langle i | \otimes \mathbb{1}_n) \\
&= \text{tr}(\langle i | \otimes U_i) \rho_l^C (| i \rangle \otimes U_j^\dagger) \\
&= \text{tr}(| i \rangle \langle i | \otimes \mathbb{1}_n) \rho_l^C \\
&= \text{tr}(| i \rangle \langle i | \text{tr}_P(\rho_l^C)) = \frac{1}{3},
\end{aligned} \tag{32}$$

where the first equality is from Eq. (21) and the third and the sixth equalities are from Eq. (22). The above results imply that for $l + 1$ steps the following holds

$$\langle i | \text{tr}_P(\rho_{l+1}) | i \rangle = \frac{1}{3} \quad \forall i \in \{0, 1, 2\}. \tag{33}$$

Next we will try to demonstrate that the property from Eq. (25) holds for the state after $l + 1$ steps. First, after the application of the coin operator we have

$$\begin{aligned}
\langle i, e_m | \rho_l^C | i, e_m \rangle &= \langle i, e_m | (C \otimes \mathbb{1}_n) \rho_l (C^\dagger \otimes \mathbb{1}_n) | i, e_m \rangle \\
&= \sum_{k, k'} \langle i | C | k \rangle \langle k, e_m | \rho_l | k', e_m \rangle \langle k' | C^\dagger | i \rangle \\
&= a_m \sum_k |c_{i,k}|^2 = a_m,
\end{aligned} \tag{34}$$

and for the shift operator

$$\begin{aligned}
\langle i, e_m | S \rho_l^C S^\dagger | i, e_m \rangle &= \langle i, e_m | \sum_c (|c\rangle\langle c| \otimes U_c) \rho_l^C \sum_c (|c\rangle\langle c| \otimes U_c^\dagger) | i, e_m \rangle \\
&= e^{i\alpha_{i,m}} \langle i, e_m | \rho_l^C | i, e_m \rangle e^{-i\alpha_{i,m}} \\
&= \langle i, e_m | \rho_l^C | i, e_m \rangle = a_m,
\end{aligned} \tag{35}$$

where the third equality is derived from the fact that $|e_m\rangle$ are eigenvectors of matrices U_c . The above result proves that the property from assumption (25) holds for $l + 1$ steps.

Furthermore, we show the inductive step for the expression (24). For a coin operator from the assumption (25) and for $i \neq j$

$$\begin{aligned}\langle i, e_m | \rho_l^C | j, e_m \rangle &= \langle i, e_m | (C \otimes \mathbb{1}_n) \rho_l (C^\dagger \otimes \mathbb{1}_n) | j, e_m \rangle \\ &= \sum_{k, k'} (\langle i | C | k \rangle \langle k, e_m |) \rho(\langle k' | C^\dagger | j \rangle | k', e_m \rangle) \\ &= a_m \sum_k \langle i | C | k \rangle \langle k | C^\dagger | j \rangle = 0,\end{aligned}\tag{36}$$

and for the shift operator, from Eq. (35) and from the assumptions we obtain

$$\begin{aligned}\langle i, e_m | S \rho_l^C S^\dagger | j, e_m \rangle &= \langle i, e_m | \sum_c (|c\rangle \langle c| \otimes U_c) \rho_l^C \sum_c (|c\rangle \langle c| \otimes U_c^\dagger) | j, e_m \rangle \\ &= e^{i\alpha_{i,m}} \langle j, e_m | \rho_l^C | j, e_m \rangle e^{-i\alpha_{j,m}} \\ &= e^{i(\alpha_{i,m} - \alpha_{j,m})} \langle i, e_m | \rho_l^C | j, e_m \rangle = 0.\end{aligned}\tag{37}$$

This proves that the property from expression (24) holds for $l + 1$ steps. Finally, we make the inductive step for off-diagonal elements in expression (23).

Let us denote the state after l evolutions and the application of the coin operator of the form

$$\rho_l^C = \frac{1}{3} \sum_k |\psi_k\rangle \langle \psi_k|. \tag{38}$$

From Eq. (31) we know that off-diagonal elements in partial trace on ρ_l^C are equal to zero. Now we will demonstrate this after the application of the shift operator. We have

$$\begin{aligned}\langle i | \text{tr}_P S \rho_l^C S^\dagger | j \rangle &= \langle i | \text{tr}_P (\mathbb{1}_3 \otimes U_i) \rho_l^C (\mathbb{1}_3 \otimes U_j^\dagger) | j \rangle \\ &= \text{tr}(\mathbb{1}_3 \otimes U_i) \rho_l^C (\mathbb{1}_3 \otimes U_j^\dagger) (|j\rangle \langle i| \otimes \mathbb{1}_n) \\ &= \sum_k \text{tr}(\langle i | \otimes U_i) |\psi_k\rangle \langle \psi_k| (|j\rangle \otimes U_j^\dagger) \\ &= \sum_{k,m} e^{-i\alpha_{j,m}} e^{i\alpha_{i,m}} \langle \psi_k | (|j\rangle \langle i| \otimes |e_m\rangle \langle e_m|) | \psi_k \rangle \\ &= \sum_m e^{-i\alpha_{j,m}} e^{i\alpha_{i,m}} \langle i, e_m | \rho_l^C | j, e_m \rangle = 0,\end{aligned}\tag{39}$$

where we have used Eqs. (21), (22) and (38).

Using the above with the result in Eq. (33) we can prove that after $l + 1$ steps

$$\text{tr}_P(\rho_{l+1}) = \frac{\mathbb{1}_3}{3}. \tag{40}$$

The above result implies that after an arbitrary number of steps, a quantum walker moves right, left or jumps with equal probabilities.

Let C^{ρ_l} be a random variable corresponding to the measurement on the coin register after l steps. We can analyse the probability distribution of the outcomes on the position register

after l steps in the terms of a random variable X^{ρ_l} . We have

$$\begin{aligned}
E(X^{\rho_{l+1}}) &= \frac{1}{3}E(X^{\rho_{l+1}}|c=0) + \frac{1}{3}E(X^{\rho_{l+1}}|c=1) + \frac{1}{3}E(X^{\rho_{l+1}}|c=2) \\
&= \frac{1}{3}\left(E(X^{\rho_l^C}|c=0) - 1\right) + \frac{1}{3}\left(E(X^{\rho_l^C}|c=1) + 1\right) + \frac{1}{3}\left(E(X^{\rho_l^C}|c=2) + a\right) \\
&= \frac{1}{3}\left(E(X^{\rho_l^C}|c=0) + E(X^{\rho_l^C}|c=1) + E(X^{\rho_l^C}|c=2)\right) + \frac{1}{3}(-1 + 1 + a) \\
&= E(X^{\rho_l^C}) + \frac{1}{3}(-1 + 1 + a) \\
&= E(X^{\rho_l}) + \frac{a}{3},
\end{aligned} \tag{41}$$

where $P(C^{\rho_{l+1}} = i) = \frac{1}{3}$ for all $i \in \{0, 1, 2\}$. This proves the claim. \blacksquare

4.2 Detecting broken links

As the second application of the introduced model we describe a simple method of detecting link failures in the network.

Let us consider the network delivering the connection for the implementation of the lively quantum walk. In such situation the limiting distribution will have the properties described in Section 3.

Let us now consider a failure of the network, which can be described as a lack of one of the links. (see Fig. 2(a)). In this case the *broken link* can be understood as a connection error or as an action of a malicious party.

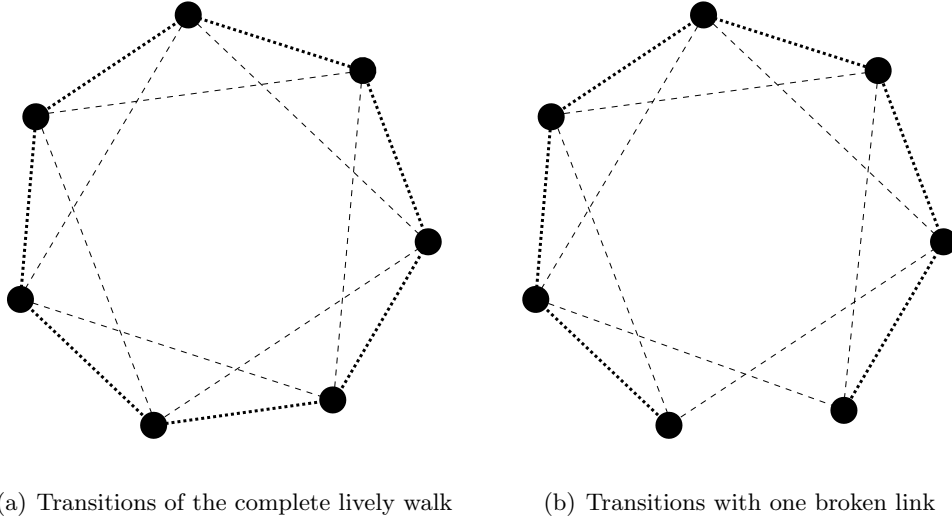


Figure 2: Illustration of the simple network with one broken link for the lively walk with $n = 7$ nodes and $a = 2$. For the standard quantum walker the broken cycle in Fig. (b) is equivalent to the line segment.

We can define the model used to describe a lively walk on a cycle with one broken link. We assume that the walker is able to execute moves with $a = 2$. In such case the shift operator is

given as

$$S_B^{n,2} = \sum_{x=1}^{n-2} S_x^{n,2} + |1\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 0| + |2\rangle\langle 2| \otimes |2\rangle\langle 0| \\ + |0\rangle\langle 1| \otimes |n-1\rangle\langle n-1| + |0\rangle\langle 0| \otimes |n-2\rangle\langle n-1| + |2\rangle\langle 2| \otimes |0\rangle\langle n-2| \quad (42)$$

The above can be interpreted as laziness condition – *i.e.* at the nodes with the broken links the walker does not move if the coin state indicates the step in the direction of the broken link. However, one should note that due to the possibility of executing steps with a larger range, the cycle with broken links is not equivalent to a line segment.

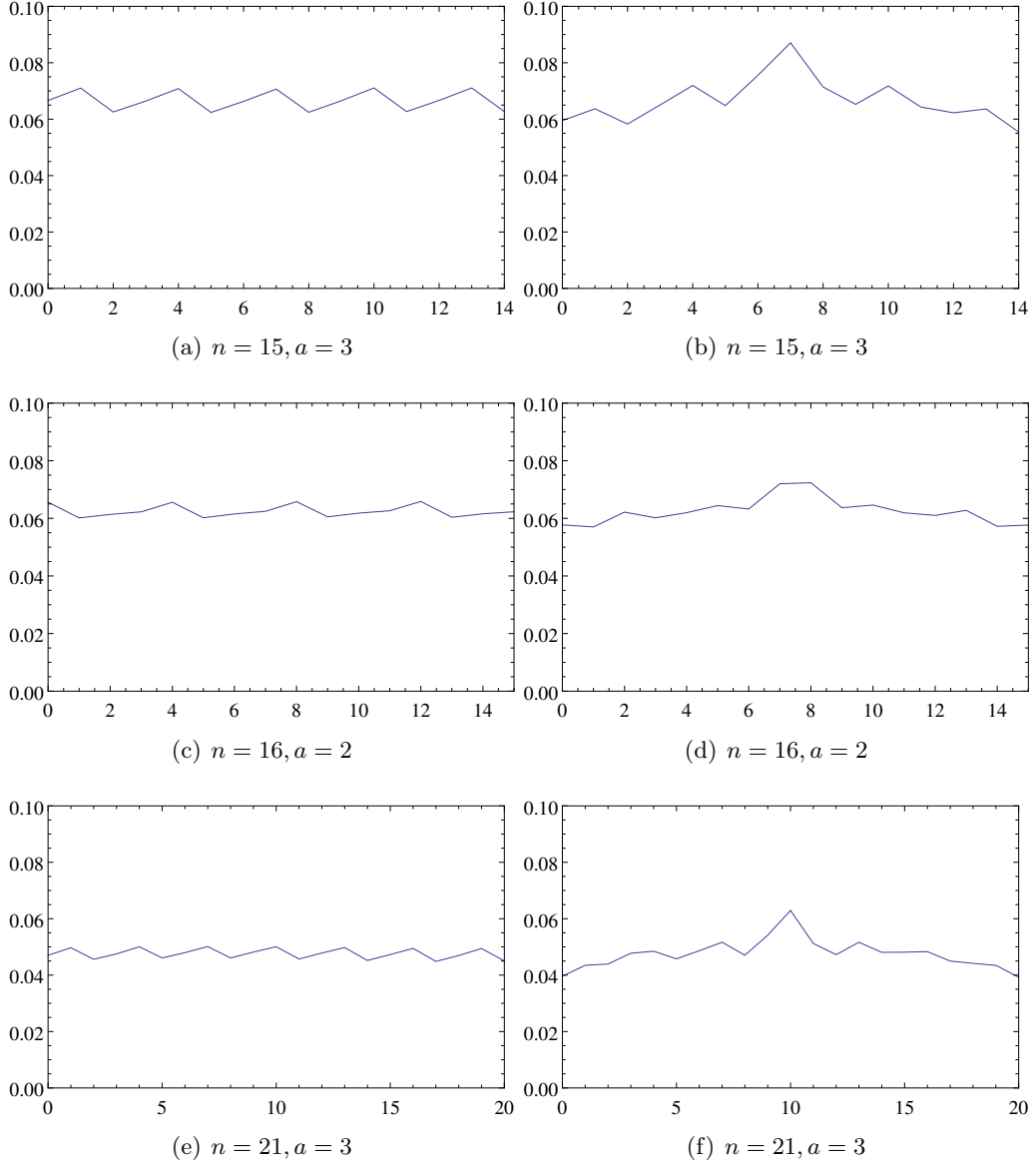


Figure 3: Influence of one broken link on the time-averaged limiting probability distribution. Plots (a), (c) and (e) illustrate the distribution without broken links, whilst plots (b), (d) and (f) provide the distribution for the walk executed on a system with one broken link.

The time-averaged limiting distribution in this situation is presented in Fig. 3. One can

easily observe that the situation where one of the links is missing has a tremendous impact on the periodicity of the limiting distribution.

One should note that this enables us to detect the malicious activity in the network easily. Moreover, this can be achieved without the full knowledge of the limiting distribution. Actually, one can detect the broken link by probing the limiting distribution in the appropriately chosen positions only.

5 Concluding remarks

In this work we have introduced a parametrized model of a quantum walk on cycle, which can be used to study the situation where the near-neighbor communication in the network is supplemented by the existence of the long-range links between the selected nodes.

We have studied the limiting behavior of the introduced model. We have proved that the periodicity of the limiting distribution is connected to the liveliness parameter.

We have also argued that the presented model can be useful for the purpose of developing new protocols for the exploration of quantum networks. We have shown that the utilization of the introduced model allows avoiding the trapping of the walk (the expectation value of the difference of position is non-zero). We have also demonstrated that the symmetry of the time-averaged limiting distribution vanishes in the situation where one of the links does not work properly. This suggests that the application of lively quantum walks can be beneficial from the point of view of quantum network exploration.

One of the interesting properties of the introduced model is that it can be used to fix the degree of connectivity of the network regarding the liveliness parameter. As such it can be used as a tool for analyzing the impact of the connectivity in the network on the properties of quantum walks.

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